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PHYSICS HELPING MATHEMATICS: A COUPLE OF EXAMPLES

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Abstract. We present two examples in analysis (complex and real) that illustrate the utility of physical intuition in mathematical proofs. The examples are related to the Riemann mapping theorem in complex analysis and the multivariable calculus question of which curl-free vector fields are gradients of functions.

1. Introduction

The programme of making physics rigorous is called mathematical physics (one of Hilbert's problems). The interaction between these two subjects works in the other direction too. That is, there are purely mathematical problems (that a priori have no connection with physics) which can be solved using intuition coming from physics. The modern aspects of this interaction are dubbed "physical mathematics" [3]. There are several examples of such fruitful synergies between the subjects. In this article, we will present two relatively elementary (advanced undergraduate or early graduate level) topics that illustrate the utility of physical intuition in mathematics.

2. The Riemann mapping theorem and electrostatics

The Riemann mapping theorem states the following.

Theorem 2.1. *Let $U \subset \mathbb{C}$ be a simply connected open set. There exists an invertible analytic map $f : U \to \mathbb{D}$ (where $\mathbb{D}$ is the unit disc centred at the origin) such that its inverse is also analytic. If there are two such maps, they differ by a Möbius transformation of the disc to itself.*

The uniqueness part follows from standard complex analysis (Schwarz lemma). This part was proven by Poincaré much later. Riemann's original

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formulation of the existence part of the theorem assumed that the subset U was bounded and had a smooth boundary, i.e., near every boundary point, the boundary could be parametrised as a smooth regular path. We shall prove the theorem under this assumption. The physical idea at play is to pretend that U is an insulator whose boundary is grounded, i.e., has zero electric potential. Then we place a charge at a point $p \in U$. If U is a disc, then choose p to be the centre. In that case, the equipotential curves are concentric circles and the electric field lines radiate outwards. We expect a similar phenonmenon to take place when U is not the unit disc. The corresponding Riemann map is simply the map that takes the equipotential curves to the circles and the field lines to the radial lines. The catch with this "proof" is that we need to rigorously prove that the potential due to a charge with grounded boundary exists. Riemann assumed this fact without proof [1]. (This problem can be phrased as finding the minimum of the electrostatic energy, and it was assumed back in the day that this minimum exists. This fact was termed 'Dirichlet's principle' by Riemann.)

The details of Riemann's proof are as follows.

(1) Existence of a Green's function (potential of a unit charge): There exists a smooth solution u on $\bar{U}$ satisfying $\Delta u = 0$ on U and $u(z) = \ln|z-p|$ on ∂U. This fact is deep and its proof will take us too far afield. We will give a sketch of the proof later. Consider $G = \ln|z-p| - u$. This function is smooth away from p, $\Delta G = 0$ away from p, $G = 0$ on the boundary, and G corresponds to the potential with a "unit charge" at p. (Rigorously, $\Delta G = \delta(p)$ where δ is the Dirac delta distribution.)

(2) Existence of a harmonic conjugate (a function whose level sets are field lines): There exists a smooth function v on U such that $\nabla v = (-u_y, u_x)$ (that is, its gradient points along the equipotential curves). The function is defined as $v(z) = \int_p^z (-u_y, u_x).\vec{dl}$ along any path $\alpha(t)$ from p to z. The fact that this integral is path-independent follows from simple-connectedness. Indeed, suppose $\alpha(t), \beta(t)$ are two such paths, then consider the piecewise smooth closed path $\gamma(t) : [0,1] \to U$ given by the concatentation of $\alpha(2t)$ and $\beta(2-2t)$. By simple-connectedness, there is a continuous map $H(t,s) : [0,1] \times [0,1] \to U$ such that $H(1,s) = H(0,s) = p \ \forall \ s, H(t,1) = p \ \forall \ t, H(t,0) =$

$\gamma(t)\ \forall\ t$. Actually, it turns out that H can be assumed to be piecewise smooth without loss of generality (Whitney's approximation theorem). Now

$$\int_0^1 \int_0^1 \left(\frac{\partial(-u_y)}{\partial y} \left(-\frac{\partial y}{\partial t}\frac{\partial x}{\partial s} + \frac{\partial x}{\partial t}\frac{\partial y}{\partial s} \right) + \frac{\partial u_x}{\partial x} \left(-\frac{\partial y}{\partial s}\frac{\partial x}{\partial t} + \frac{\partial y}{\partial t}\frac{\partial x}{\partial s} \right) \right) dsdt = 0$$

by the harmonicity of u. By the fundamental theorem of calculus, this integral is also equal to $\int \vec{F}.\vec{dl}(s=1) - \int \vec{F}.\vec{dl}(s=0)$ where $\vec{F} = (-u_y, u_x)$. (We have essentially proven Green's theorem in this special case.)

The above argument shows that v is well-defined. It is easy to check that $u + \sqrt{-1}v$ is a holomorphic function (that is, the Cauchy-Riemann equations are met).

(3) Construction of the map: Consider $f(z) = e^{u(z)+\sqrt{-1}v(z)}(z-p)$. This function is holomorphic and takes p to 0. Note that $|f| = |z-p|e^{u(z)} = e^{G(z)}$ and $Arg(f(z)) = v + Arg(z-p)$. As expected, the "equipotential" curves are taken to circles and the "field lines" to radial lines. By the mean value property for harmonic functions, no local maxium of G is attained. Proving that f is indeed a biholomorphism taking Ω to $\mathbb{D}$ is more technical.

(4) Proof that the map actually works: On the boundary, i.e., $\partial\Omega \cup \{p\}$, $G \leq 0$. Hence, $G \leq 0$ throughout. This means that $f(\Omega) \subset \mathbb{D}$. By the open mapping theorem for holomorphic functions, the image $f(\Omega)$ is open. We claim that it is closed too. Suppose $f(z_n) \to b \in \mathbb{D}$. By compactness of $\bar{\mathbb{D}}$, a subsequence z_{n_k} converges to some $z \in \bar{\mathbb{D}}$. As $z_n \in \Omega \to \partial\Omega$, we see that $|f(z_n)| \to 1$. Since $|b| < 1$, we see that $z \in \mathbb{D}$ and $f(z) = b$. Thus f is onto. Moreover, $f^{-1}(p) = \{0\}$, and since $f'(p) > 0$ we see that the multiplicity of the root at p is 1. By the argument principle of complex analysis, we see that f is $1-1$. A holomorphic $1-1$ onto function is a biholomorphism (again by the argument principle).

As mentioned earlier, the solvability of $\Delta u = 0$ with Dirichlet boundary conditions is a tricky affair (that Riemann assumed without proof). While we shall not attempt to prove it in this article, what follows is a very high-level sketch of the ideas involved. By means of extending the boundary function to a smooth function and subtraction, we can reduce this problem to solving $\Delta u = f$ with $u = 0$ on the boundary. The strategy is to consider the "electrostatic energy" $E[u] = \int_{\Omega} (|\nabla u|^2 + fu)$ and attempt to minimise it (with the restriction that u is zero on the boundary). If a smooth minimiser u_0 does exist, then $\frac{dE[u_0+tv]}{dt}|_{t=0} = 0$ for all smooth functions v whose restriction to the boundary is zero. One can use this fact to conclude that u solves $\Delta u = f$. To minimise this energy, one needs a string of inequalities. One first proves that E is bounded below on the space of smooth compactly supported functions in Ω. Then one considers a sequence of functions converging to the infimum of E. The difficulty is that, a priori, no subsequence of such a sequence necessarily converges (in a reasonable sense) to a smooth function. One proves that convergence does happen in some sense to a potentially non-smooth function. Then one proves that the non-smooth function is actually smooth and hence solves the desired equation. □

This approach of proving results in complex analysis motivated from electrostatics has taken a life of its own (potential theory). In fact, one can use these ideas to generalise the Riemann mapping theorem to the famous uniformisation theorem of Riemann surfaces.

3. Shapes and electromagnetic fields

The line integrals of conservative vector fields $\vec{F}$ do not depend on the paths taken. Such vector fields are gradients of potential energy functions, i.e., $\vec{F} = \nabla U$. In particular, $\nabla \times \vec{F} = \vec{0}$. One might wonder if the converse is true. It can be easily shown to be true if the domain of $\vec{F}$ is all of $\mathbb{R}^2$ (by simply finding U using the work done by $\vec{F}$ along a straight line).

Using Ampère's law applied to an infinite current carrying wire perpendicular to the plane (and passing through the origin), we see that the magnetic field is $\vec{B} = (-\frac{y}{x^2+y^2}, \frac{x}{x^2+y^2})$. The domain of this field is $\mathbb{R}^2 - \{(0,0)\}$. This field obeys Maxwell's equation: $\nabla \times \vec{B} = \vec{0}$ on its domain. Unfortunately, $\vec{B} \neq \nabla U$. (If it were, then the line integral around a circle ought to have been 0 but it can be calculated to be 2π.) However, interestingly enough, this is the only thing that can go wrong for a smooth vector field $\vec{F}$ in $\mathbb{R}^2 - \{(0,0)\}$. The proof of this fact is technical but a sketch is as follows: Suppose $\nabla \times \vec{F} = \vec{0}$. Then let $c = \int \vec{F}.d\vec{l}$ around the unit circle anticlockwise, and $\vec{G} = \vec{F} - \frac{c}{2\pi}\vec{B}$. Now the line integral of $\vec{G}$ along the unit circle is 0. We simply need to prove that the line integral along any other piecewise smooth simple closed curve is 0. (Then we can simply find the work done and produce a potential.) Firstly, the integral along *any* circle centred at the origin is 0 by Green's theorem. Secondly, any piecewise smooth simple closed curve that does not enclose the origin (recall that the deep Jordan curve theorem asserts that any simple closed curve divides the plane into two parts) leads to a zero line integral by Green's theorem. If it does enclose the origin, then simply throw out a disc around the origin and use Green's theorem to conclude that the line integral is the same as that over a small circle, which is zero. (The cleanest way of making this sketch somewhat self-contained is to simply prove that the line integral is invariant under piecewise smooth deformations of curves akin to 2, and to prove that any piecewise smooth simple closed path can be deformed to a constant path if it does not enclose the origin, and to a circle if it does. The latter can be proven by approximating using piecewise-linear approximations.)

In fact, the arguments above can be generalised to show that if we throw out n points from the plane, then after subtracting multiples of the magnetic fields of n current carrying wire, a field $\vec{F}$ satisfying $\nabla \times \vec{F} = \vec{0}$ is conservative. In other words, there is an n-dimensional vector space worth of things that can go wrong, i.e., curl-free fields upto gradients of potential energies. Therefore, we can glean information about the shape of a domain (how many holes it has) by simply answering the question of which vector fields are conservative.

This philosophy of relating the shape of an object to the solutions of certain partial differential equations (arising from physics) on it is very general. Donaldson [2] used it to great effect to prove that there are different ways of "doing calculus" on $\mathbb{R}^4$. (More precisely, that there is a non-standard smooth manifold structure on $\mathbb{R}^4$.) Later on, Seiberg and Witten [4] simplified Donaldson's proof by using another set of equations arising from physics.

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References

[1] L. Ahlfors, E. Calabi, M. Morse, L. Sario, and D. Spencer (eds.), Developments of the Theory of Conformal Mapping and Riemann Surfaces Through a Century, *Contributions to the Theory of Riemann Surfaces*: 3–4 (1953).

[2] S. Donaldson, An Application of Gauge Theory to Four Dimensional Topology, *Journal of Differential Geometry* **18**(2) (1983): 279–315.

[3] GW. Moore, Physical mathematics and the future. InVision talk, Strings conference, Princeton 2014 (Vol. 10).

[4] N. Seiberg and E. Witten, Electric-magnetic duality, monopole condensation, and confinement in N=2 supersymmetric Yang-Mills theory, *Nuclear Physics B.* **426**(1) (1994): 19–52.

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